

On the range of the Douglas–Rachford operator

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Abstract

The problem of finding a minimizer of the sum of two convex functions — or, more generally, that of finding a zero of the sum of two maximally monotone operators — is of central importance in variational analysis. Perhaps the most popular method of solving this problem is the Douglas–Rachford splitting method. Surprisingly, little is known about the range of the Douglas–Rachford operator.

In this paper, we set out to study this range systematically. We prove that for 3* monotone operators a very pleasing formula can be found that reveals the range to be nearly equal to a simple set involving the domains and ranges of the underlying operators. A similar formula holds for the range of the corresponding displacement mapping. We discuss applications to subdifferential operators, to the infimal displacement vector, and to firmly nonexpansive mappings. Various examples and counter-examples are presented, including some concerning the celebrated Brezis–Haraux theorem.

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1 Introduction

Unless otherwise stated, throughout this paper

X is a finite-dimensional real Hilbert space

with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. Let $A : X \rightrightarrows X$ be a set-valued operator. We say that A is *monotone* if $\langle x - y, u - v \rangle \geq 0$ for all pairs (x, u) and (y, v) in $\text{gra } A$, the *graph* of A . A monotone operator A is *maximally monotone* if the graph of A , denoted $\text{gra } A$, cannot be properly extended without destroying the monotonicity of A . Monotone operators are of considerable importance in optimization and variational analysis; see, e.g., [5], [10], [12], [23], [24], [26], [27], [28], [30], [31], and [32]. It is well known that the subdifferential operator of a proper lower semicontinuous convex function is maximally monotone. Subdifferential operators also belong to the class of 3^* monotone (also known as rectangular) operators which was introduced by Brezis and Haraux [11]; see also [3] and [4]. The sum of two maximally monotone operators is monotone; maximality, however, is guaranteed only in presence of a constraint qualification [25]. The problem of finding the zeros of the sum of two maximally monotone operators is an active topic in optimization as it captures the key problem of minimizing a sum of two convex functions. More broadly, from an optimization perspective, constrained optimization problems, convex feasibility problem as well as many other optimization problems can be interpreted and recast as the problem of finding the zeros of the sum of two maximally monotone operators. Most methods for solving the sum problem are splitting algorithms; the most popular of which is the celebrated Douglas–Rachford method, which was adopted to the monotone operator framework by Lions and Mercier [19]. (See also e.g. [5], [14], [15], [16], and [18] for further results on and applications of this algorithm.)

Let $A : X \rightrightarrows X$ be maximally monotone. Recall that the *resolvent* of A is $J_A = (\text{Id} + A)^{-1}$, where Id denotes the identity operator. Moreover, if A is maximally monotone, then J_A is single-valued, firmly nonexpansive, and maximally monotone. The *reflected resolvent* of A is $R_A = 2J_A - \text{Id}$. Now let $B : X \rightrightarrows X$ be also maximally monotone. The *Douglas–Rachford splitting operator* for the pair (A, B) is

$$(1) \quad T_{(A,B)} := \frac{1}{2} \text{Id} + \frac{1}{2} R_B R_A = \text{Id} - J_A + J_B R_A.$$

One main goal of this work is to analyze the ranges of $T_{(A,B)}$ and of the *displacement mapping* $\text{Id} - T_{(A,B)}$. It is known (see, e.g., [2, Corollary 2.14] or [17, Proposition 4.1]) that

$$(2) \quad \text{ran}(\text{Id} - T_{(A,B)}) \subseteq (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B).$$

It is natural to inquire whether or not this is a mere inclusion or perhaps even an inequality. In general, this inclusion is strict — sometimes even extremely so in the sense that

$\text{ran}(\text{Id} - T_{(A,B)})$ may be a singleton while $(\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B)$ may be the entire space; see Example 4.9. This likely has discouraged efforts to obtain a better description of these ranges. However, and somewhat surprisingly, we are able to obtain — under fairly mild assumptions on A and B — the simple and elegant formulae

$$(3) \quad \text{ran}(\text{Id} - T_{(A,B)}) \simeq (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B)$$

and

$$(4) \quad \text{ran } T_{(A,B)} \simeq (\text{dom } A - \text{ran } B) \cap (\text{ran } A + \text{dom } B),$$

where the “near equality” of two sets C and D , denoted by $C \simeq D$, means that the two sets have the same relative interior (the interior with respect to the closed affine hull) and the same closure. When $A = \partial f$ and $B = \partial g$ are subdifferential operators, which is the key setting in convex optimization, the above formulae can be written as

$$(5) \quad \text{ran}(\text{Id} - T_{(\partial f, \partial g)}) \simeq (\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*)$$

and

$$(6) \quad \text{ran } T_{(\partial f, \partial g)} \simeq (\text{dom } f - \text{dom } g^*) \cap (\text{dom } f^* + \text{dom } g).$$

These results are interesting because the problem of finding a point in $(A + B)^{-1}(0)$ has a solution if and only if $0 \in \text{ran}(\text{Id} - T_{(A,B)})$ (see, e.g., [13, Lemma 2.6(ii)]). It also provides information on finding the infimal displacement vector that defines the normal problem recently introduced in [2]. Moreover, $\text{ran } T_{(A,B)}$ contains the set of fixed points of $T_{(A,B)}$. Using the correspondence between maximally monotone operators and firmly nonexpansive mappings (see Fact 2.5), we are able to reformulate our results for firmly nonexpansive mappings. In addition to our main results, we show that the well-known conclusion of Brezis-Haraux Theorem [11] is optimal in the sense that actual equality may fail (see Example 3.14 and Proposition 7.4). Our investigation relies on the class of 3^* monotone operators (see [11]), Attouch-Théra duality (see [1]), and the associated normal problem (see [2]).

The remainder of this paper is organized as follows. Section 2 contains a brief collection of facts on monotone operators and their resolvents, as well as on firmly nonexpansive mappings. In Section 3, we review the notions of near convexity and near equality, and we also present some new results. Section 4 is concerned with the Attouch-Théra duality, the normal problem, and the Douglas-Rachford splitting operator. Our main results are presented in Section 5, while applications and special cases are provided in Section 6. In Section 7, we offer some results that are valid in a possibly infinite-dimensional Hilbert space. We also provide various examples and counterexamples.

We conclude this section with some comments on notation. We use P_C and N_C to denote projector and the normal cone operator associated with the nonempty closed convex subset C of X . The recession cone of C is $\text{rec } C := \{x \in X \mid x + C \subseteq C\}$, and the polar cone of C is $C^\ominus := \{u \in X \mid \sup \langle C, u \rangle \leq 0\}$. We use $\text{ball}(x; r)$ to denote the closed ball in X centred at $x \in X$ with radius $r > 0$. For a subset S of X , the relative interior of the set S is $\text{ri } S := \{s \in S \mid (\exists r > 0) \text{ball}(s; r) \cap \overline{\text{aff } S} \subseteq S\}$, where $\text{aff } S$ denotes the affine hull of S . All other notation is standard and follows, e.g., [5].

2 Monotone operators and firmly nonexpansive mappings

In this short section, we review some key results on monotone operators and firmly nonexpansive mappings that are needed subsequently. (See also [5] for further results.)

Fact 2.1 (See, e.g., [24, Corollary 12.44].) *Let $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ be maximally monotone such that $\text{ri dom } A \cap \text{ri dom } B \neq \emptyset$. Then $A + B$ is maximally monotone.*

Fact 2.2 (See, e.g., [5, Proposition 23.2(i)].) *Let $A : X \rightrightarrows X$ be maximally monotone. Then $\text{dom } A = \text{ran } J_A$.*

Recall the inverse resolvent identity (see, e.g., [24, Lemma 12.14])

$$(7) \quad J_A + J_{A^{-1}} = \text{Id}.$$

Applying Fact 2.2 to A^{-1} and using (7), we obtain

$$(8) \quad \text{ran } A = \text{dom } A^{-1} = \text{ran } J_{A^{-1}} = \text{ran}(\text{Id} - J_A).$$

Fact 2.3 (Minty parametrization) (See [20].) *Let $A : X \rightrightarrows X$ be maximally monotone. Then $\text{gra } A \rightarrow X : (x, u) \rightarrow x + u$ is a continuous bijection, with continuous inverse $x \mapsto (J_A x, x - J_A x)$, and*

$$(9) \quad \text{gra } A = \{(J_A x, x - J_A x) \mid x \in X\}.$$

Fact 2.4 (See [29, Theorem 3.1].) *Let C be a nonempty closed convex subset of X . Then $\overline{\text{ran}}(\text{Id} - P_C) = (\text{rec } C)^\ominus$.*

Let $T : X \rightarrow X$. Recall that T is *nonexpansive* if $(\forall x \in X)(\forall y \in X) \|Tx - Ty\| \leq \|x - y\|$, and T is *firmly nonexpansive* if

$$(10) \quad (\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\|^2 + \|(\text{Id} - T)x - (\text{Id} - T)y\|^2 \leq \|x - y\|^2.$$

Fact 2.5 (See, e.g., [18, Theorem 2].) *Let D be a nonempty subset of X , let $T : D \rightarrow X$, let $A : X \rightrightarrows X$, and suppose that $A = T^{-1} - \text{Id}$. Then the following hold:*

- (i) $T = J_A$.
- (ii) A is monotone if and only if T is firmly nonexpansive.
- (iii) A is maximally monotone if and only if T is firmly nonexpansive and $D = X$.

3 Near convexity and near equality

We now review and extend results on near equality and near convexity.

Definition 3.1 (near convexity) (See Rockafellar and Wets's [24, Theorem 12.41].) *Let D be a subset of X . Then D is nearly convex if there exists a convex subset C of X such that $C \subseteq D \subseteq \overline{C}$.*

Fact 3.2 (See [24, Theorem 12.41].) *Let $A : X \rightrightarrows X$ be maximally monotone. Then $\text{dom } A$ and $\text{ran } A$ are nearly convex.*

Definition 3.3 (near equality) (See [8, Definition 2.3].) *Let C and D be subsets of X . We say that C and D are nearly equal if*

$$(11) \quad C \simeq D \iff \overline{C} = \overline{D} \text{ and } \text{ri } C = \text{ri } D.$$

Fact 3.4 (See [8, Lemma 2.7].) *Let D be a nonempty nearly convex subset of X , say $C \subseteq D \subseteq \overline{C}$, where C is a convex subset of X . Then*

$$(12) \quad D \simeq \overline{D} \simeq \text{ri } D \simeq \text{conv } D \simeq \text{ri conv } D \simeq C.$$

In particular, \overline{D} and $\text{ri } D$ are convex and nonempty.

Fact 3.5 (See [8, Proposition 2.12(i)&(ii)].) *Let C and D be nearly convex subsets of X . Then*

$$(13) \quad C \simeq D \iff \overline{C} = \overline{D}.$$

Fact 3.6 (See [8, Lemma 2.13].) *Let $(C_i)_{i \in I}$ be a finite family of nearly convex subsets of X , and let $(\lambda_i)_{i \in I}$ be a finite family of real numbers. Then $\sum_{i \in I} \lambda_i C_i$ is nearly convex and $\text{ri}(\sum_{i \in I} \lambda_i C_i) = \sum_{i \in I} \lambda_i \text{ri } C_i$.*

Fact 3.7 (See [8, Theorem 2.14].) *Let $(C_i)_{i \in I}$ be a finite family of nearly convex subsets of X , and let $(D_i)_{i \in I}$ be a family of subsets of X such that $C_i \simeq D_i$ for every $i \in I$. Then $\sum_{i \in I} C_i \simeq \sum_{i \in I} D_i$.*

Fact 3.8 (See [23, Theorem 6.5].) Let $(C_i)_{i \in I}$ be a finite family of convex subsets of X . Suppose that $\bigcap_{i \in I} \text{ri } C_i \neq \emptyset$. Then $\overline{\bigcap_{i \in I} C_i} = \bigcap_{i \in I} \overline{C_i}$ and $\text{ri } \bigcap_{i \in I} C_i = \bigcap_{i \in I} \text{ri } C_i$.

Most of the following results are known. For different proofs see also [9, Theorem 2.1] and the forthcoming [21] and [22].

Lemma 3.9 Let C and D be nearly convex subsets of X such that $\text{ri } C \cap \text{ri } D \neq \emptyset$. Then the following hold:

- (i) $C \cap D$ is nearly convex.
- (ii) $C \cap D \simeq \text{ri } C \cap \text{ri } D$.
- (iii) $\text{ri}(C \cap D) = \text{ri } C \cap \text{ri } D$.
- (iv) $\overline{C \cap D} = \overline{C} \cap \overline{D}$.

Proof. (i): Since C and D are nearly convex, by Fact 3.4, $\text{ri } C$ and $\text{ri } D$ are convex. Consequently,

$$(14) \quad \text{ri } C \cap \text{ri } D \text{ is convex,}$$

and clearly

$$(15) \quad \text{ri } C \cap \text{ri } D \subseteq C \cap D.$$

By Fact 3.4 we have $\text{ri } C \simeq C$ and $\text{ri } D \simeq D$. Hence, $\text{ri}(\text{ri } C) = \text{ri } C$ and $\text{ri}(\text{ri } D) = \text{ri } D$. Therefore,

$$(16) \quad \text{ri}(\text{ri } C) \cap \text{ri}(\text{ri } D) = \text{ri } C \cap \text{ri } D \neq \emptyset.$$

Using (16) and Fact 3.8 applied to the convex sets $\text{ri } C$ and $\text{ri } D$ yield $\text{ri}(C \cap D) = \text{ri } C \cap \text{ri } D$; hence

$$(17) \quad \overline{\text{ri } C \cap \text{ri } D} = \overline{\text{ri } C} \cap \overline{\text{ri } D}.$$

Since $\text{ri } C \simeq C$ and $\text{ri } D \simeq D$ by Fact 3.4, we have $\overline{\text{ri } C} = \overline{C}$ and $\overline{\text{ri } D} = \overline{D}$. Combining with (15) and (17), we obtain

$$(18) \quad \text{ri } C \cap \text{ri } D \subseteq C \cap D \subseteq \overline{C} \cap \overline{D} = \overline{\text{ri } C} \cap \overline{\text{ri } D} = \overline{\text{ri } C \cap \text{ri } D},$$

which in turn yields (i) in view of (14). (ii): Use (i) and Fact 3.4 applied to the convex set $\text{ri } C \cap \text{ri } D$ and the nearly convex set $C \cap D$. (iii): Using (ii), Fact 3.8 applied to the convex sets $\text{ri } C$ and $\text{ri } D$ and (16) we have

$$(19) \quad \text{ri}(C \cap D) = \text{ri}(\text{ri } C \cap \text{ri } D) = \text{ri}(\text{ri } C) \cap \text{ri}(\text{ri } D) = \text{ri } C \cap \text{ri } D,$$

as required. (iv): Since C and D are nearly convex, it follows from Fact 3.4 that $\overline{\text{ri } C} = \overline{C}$ and $\overline{\text{ri } D} = \overline{D}$. Combining with (ii) and (17) we have

$$(20) \quad \overline{C \cap D} = \overline{\text{ri } C \cap \text{ri } D} = \overline{\text{ri } C} \cap \overline{\text{ri } D} = \overline{C} \cap \overline{D},$$

as claimed. ■

Corollary 3.10 *Let C_1 and C_2 be nearly convex subsets of X , and let D_1 and D_2 be subsets of X such that $C_1 \simeq D_1$ and $C_2 \simeq D_2$. Suppose that $\text{ri } C_1 \cap \text{ri } C_2 \neq \emptyset$. Then*

$$(21) \quad C_1 \cap C_2 \simeq D_1 \cap D_2.$$

Proof. Let $i \in \{1, 2\}$. Since $C_i \simeq D_i$, by Definition 3.3 we have

$$(22) \quad \text{ri } C_i = \text{ri } D_i \quad \text{and} \quad \overline{C_i} = \overline{D_i}.$$

Hence,

$$(23) \quad \text{ri } D_1 \cap \text{ri } D_2 = \text{ri } C_1 \cap \text{ri } C_2 \neq \emptyset.$$

Moreover, since C_i is nearly convex, it follows from Fact 3.4 that $\overline{\text{ri } C_i} = \overline{C_i}$ and $\text{ri } C_i$ is convex. Therefore

$$(24) \quad \text{ri } C_i = \text{ri } D_i \subseteq D_i \subseteq \overline{D_i} = \overline{C_i} = \overline{\text{ri } C_i}.$$

Hence D_i is nearly convex. Applying Lemma 3.9 (iii) to the two sets C_1 and C_2 implies that

$$(25) \quad \text{ri}(C_1 \cap C_2) = \text{ri } C_1 \cap \text{ri } C_2.$$

Similarly we have

$$(26) \quad \text{ri}(D_1 \cap D_2) = \text{ri } D_1 \cap \text{ri } D_2.$$

Using (23) and Lemma 3.9(i), applied to the sets C_1 and C_2 we have $C_1 \cap C_2$ is nearly convex. Similarly, $D_1 \cap D_2$ is nearly convex. By Fact 3.4 $C_1 \cap C_2 \simeq \text{ri}(C_1 \cap C_2)$ and $D_1 \cap D_2 \simeq \text{ri}(D_1 \cap D_2)$. Hence $\overline{C_1 \cap C_2} = \overline{\text{ri}(C_1 \cap C_2)}$ and $\overline{D_1 \cap D_2} = \overline{\text{ri}(D_1 \cap D_2)}$. Combining with (25), (23) and (26) yield

$$(27) \quad \overline{C_1 \cap C_2} = \overline{\text{ri}(C_1 \cap C_2)} = \overline{\text{ri } C_1 \cap \text{ri } C_2} = \overline{\text{ri } D_1 \cap \text{ri } D_2} = \overline{\text{ri}(D_1 \cap D_2)} = \overline{D_1 \cap D_2}.$$

Now, Fact 3.5 applied to the nearly convex sets $C_1 \cap C_2$ and $D_1 \cap D_2$ implies that $C_1 \cap C_2 \simeq D_1 \cap D_2$. ■

Definition 3.11 (3^* monotone) (See [11, page 166].) Let $A : X \rightrightarrows X$ be monotone. Then A is 3^* monotone (also known as rectangular) if

$$(\forall x \in \text{dom } A)(\forall v \in \text{ran } A) \quad \inf_{(z,w) \in \text{gra } A} \langle x - z, v - w \rangle > -\infty.$$

Fact 3.12 (See [11, page 167].) Let $f : X \rightarrow \mathbb{R}$ be proper, convex, lower semicontinuous. Then ∂f is 3^* monotone.

Fact 3.13 (Brezis–Haraux) Let H be a real (not necessarily finite-dimensional) Hilbert space, let $A : H \rightrightarrows H$ and $B : H \rightrightarrows H$ be monotone operators such that $A + B$ is maximally monotone and one of the following conditions holds:

- (i) A and B are 3^* monotone.
- (ii) $\text{dom } A \subseteq \text{dom } B$ and B is 3^* monotone.

Then

$$(28) \quad \overline{\text{ran}}(A + B) = \overline{\text{ran } A + \text{ran } B} \quad \text{and} \quad \text{int } \text{ran}(A + B) = \text{int}(\text{ran } A + \text{ran } B).$$

If H is finite-dimensional, then

$$(29) \quad \text{ran}(A + B) \text{ is nearly convex} \quad \text{and} \quad \text{ran}(A + B) \simeq \text{ran } A + \text{ran } B.$$

Proof. See [11, Theorems 3 and 4] for the proof of (28) and [8, Theorem 3.13] for the proof of (29). ■

Example 3.14 and Proposition 7.2 illustrate that the results of Fact 3.13 are optimal in the sense that actual equality fails.

Example 3.14 Suppose that $X = \mathbb{R}^2$ and let $f : \mathbb{R}^2 \rightarrow]-\infty, +\infty]$: $(\xi_1, \xi_2) \mapsto \max\{g(\xi_1), |\xi_2|\}$, where $g(\xi_1) = 1 - \sqrt{\xi_1}$ if $\xi_1 \geq 0$, $g(\xi_1) = +\infty$ if $\xi_1 < 0$. Set $A = \partial f^*$. Then A is 3^* monotone and $2A = A + A$ is maximally monotone, yet

$$(30) \quad 2 \text{ran } A = \text{ran } 2A = \text{ran}(A + A) \subsetneq \text{ran } A + \text{ran } A.$$

Proof. First notice that by Fact 3.12 A is 3^* monotone. Moreover, since A is maximally monotone, it follows from Fact 3.4 that $\text{ri dom } A$ is nonempty and convex. Since $\text{ri dom } A \cap \text{ri dom } A = \text{ri dom } A \neq \emptyset$, Fact 2.1 implies that $A + A = 2A$ is maximally monotone. Using [5, Proposition 16.24] and [23, example on page 218] we know that $\text{ran } \partial f^* = \text{ran}(\partial f)^{-1} = \text{dom } \partial f = \{(\xi_1, \xi_2) \mid \xi_1 > 0\} \cup \{(0, \xi_2) \mid |\xi_2| \geq 1\}$ and $\text{ran}(A + A) = \{(\xi_1, \xi_2) \mid \xi_1 > 0\} \cup \{(0, \xi_2) \mid |\xi_2| \geq 2\} \neq \{(\xi_1, \xi_2) \mid \xi_1 \geq 0\} = \text{ran } A + \text{ran } A$. ■

4 Attouch–Théra duality and the normal problem

This section provides a review of the Attouch–Théra duality and the associated normal problem. From now on, we assume that

$$A : X \rightrightarrows X \text{ and } B : X \rightrightarrows X \text{ are maximally monotone.}$$

We abbreviate

$$A^\circ := (-\text{Id}) \circ A \circ (-\text{Id}), \quad A^{-\circ} := (A^{-1})^\circ = (A^\circ)^{-1},$$

and we observe that A° is maximally monotone as is $(A^{-1})^\circ = (A^\circ)^{-1}$. The *primal problem* associated with the ordered pair (A, B) is to find the zeros of $A + B$. Since A and B are maximally monotone operators, so are A^{-1} and $B^{-\circ}$. The *dual pair* of (A, B) is defined by

$$(A, B)^* := (A^{-1}, B^{-\circ}).$$

We now recall the definition of the dual problem.

Definition 4.1 ((Attouch–Théra) dual problem) *The (Attouch–Théra) dual problem for the ordered pair (A, B) is to find the set of zeros of $A^{-1} + B^{-\circ}$.*

From now on, we shall use $T_{(A,B)}$ to refer to the Douglas–Rachford splitting operator for two operators A and B , defined as

$$(31) \quad T_{(A,B)} := \frac{1}{2} \text{Id} + \frac{1}{2} R_B R_A = \text{Id} - J_A + J_B R_A.$$

Fact 4.2 (self-duality) (See [17, Lemma 3.6 on page 133] or [3, Corollary 4.3].)

$$(32) \quad T_{(A^{-1}, B^{-\circ})} = T_{(A,B)}.$$

Fact 4.3 (See [17, Proposition 4.1] or [2, Corollary 2.14].)

$$(33) \quad \text{ran}(\text{Id} - T_{(A,B)}) = \{a - b \mid (a, a^*) \in \text{gra } A, (b, b^*) \in \text{gra } B, a - b = a^* + b^*\}$$

$$(34) \quad \subseteq (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B).$$

Fact 4.4 (See [2, Proposition 2.16].) $T_{(A,B)} = \text{Id} - T_{(A, B^{-1})}$.

Let $w \in X$ be fixed. For the operator A , the *inner and outer shifts* associated with A are defined by

$$(35) \quad A_w : X \rightrightarrows X : x \mapsto A(x - w),$$

$$(36) \quad {}_w A : X \rightrightarrows X : x \mapsto Ax - w.$$

Notice that A_w and ${}_w A$ are maximally monotone, with $\text{dom } A_w = \text{dom } A + w$ and $\text{dom } {}_w A = \text{dom } A$.

Definition 4.5 (The w -perturbed problem) (See [2, Definition 3.1].) *The w -perturbed problem associated with the pair (A, B) is to determine the set of zeros*

$$(37) \quad Z_w := ({}_wA + B_w)^{-1}(0) = \{x \in X \mid w \in Ax + B(x - w)\}.$$

Fact 4.6 (See [2, Proposition 3.3].) *Let $w \in X$. Then*

$$(38) \quad Z_w \neq \emptyset \iff w \in \text{ran}(\text{Id} - T_{(A,B)}) \iff w \in \text{ran}(A + B_w).$$

Definition 4.7 (normal problem and infimal displacement vector) *The normal problem associated with the pair (A, B) is the $v_{(A,B)}$ -perturbed problem, where $v_{(A,B)}$ is the infimal displacement vector for the pair (A, B) defined as*

$$(39) \quad v_{(A,B)} := P_{\overline{\text{ran}(\text{Id} - T_{(A,B)})}}(0).$$

Fact 4.8 (See [2, Proposition 3.11].)

$$(40) \quad \|v_{(A,B)}\| = \|v_{(B,A)}\|.$$

In view of Definition 4.7 and Fact 4.8, the magnitude of the vector $v_{(A,B)}$ is actually a measure of how far the original problem is from the normal problem. This magnitude is the same for the pairs (A, B) and (B, A) .

We now explore how $\text{ran}(\text{Id} - T)$ is related to the set $(\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B)$. We will prove that when the operators A and B are “sufficiently nice”, we have

$$(41) \quad \text{ran}(\text{Id} - T_{(A,B)}) \simeq (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B).$$

In general, (41) may fail spectacularly as we will now illustrate.

Example 4.9 *Suppose that $X = \mathbb{R}^2$, and that*

$$(42) \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = -A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$(43) \quad \text{ran}(\text{Id} - T_{(A,B)}) = \{0\} \subsetneq \mathbb{R}^2 = (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B).$$

Proof. Recall that $\text{dom } A = \text{dom } B = \text{ran } A = \text{ran } B = \mathbb{R}^2$, consequently $(\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B) = \mathbb{R}^2$. On the other hand, one checks that $R_A : (x, y) \mapsto (y, -x) = B$ and $R_B : (x, y) \mapsto (-y, x) = A$. Hence $R_B R_A = \text{Id}$ and therefore $\text{Id} - T_{(A,B)} = \frac{1}{2}(\text{Id} - R_B R_A) \equiv 0$. ■

5 Main results

Upholding the notation of Section 4, we also set

$$(44) \quad D := D_{(A,B)} := \text{dom } A - \text{dom } B \quad \text{and} \quad R := R_{(A,B)} := \text{ran } A + \text{ran } B.$$

We start by proving some auxiliary results.

Lemma 5.1 *The following hold:*

- (i) *The sets D and R are nearly convex.*
- (ii) $\text{ri } D \cap \text{ri } R \neq \emptyset$.
- (iii) $D \cap R$ *is nearly convex.*
- (iv) $\text{ri}(D \cap R) = \text{ri } D \cap \text{ri } R$.
- (v) $\overline{D \cap R} = \overline{\text{ri } D \cap \text{ri } R}$.
- (vi) $\overline{D \cap R} = \overline{D} \cap \overline{R}$.

Proof. (i): Combine Fact 3.2 and Fact 3.6. (ii): Since B is maximally monotone, the Minty parametrization (9) implies that $X = \text{dom } B + \text{ran } B$. Hence by (i) and Fact 3.6

$$(45) \quad 0 \in X = \text{ri } X = \text{ri}(\text{ran } A + \text{ran } B - (\text{dom } A - \text{dom } B)) = \text{ri } R - \text{ri } D.$$

Hence, $\text{ri } D \cap \text{ri } R \neq \emptyset$, as claimed. (Note that we did not use the maximal monotonicity of A in this proof.) (iii): Combine (i), (ii) and Lemma 3.9(i). (iv): Combine (i), (ii) and Lemma 3.9(iii). (v): Combine (i), (ii) and Lemma 3.9(ii). (vi): Combine (i), (ii) and Lemma 3.9(iv). ■

Theorem 5.2 *Suppose that A and B satisfy one of the following:*

- (i) $(\forall w \in \text{ri } D \cap \text{ri } R) \quad \text{ri}(\text{ran } A + \text{ran } B) \subseteq \text{ri } \text{ran } (A + B_w)$.
- (ii) A and B are 3^* monotone.
- (iii) $\text{dom } B + \text{ri } D \cap \text{ri } R \subseteq \text{dom } A$ and A is 3^* monotone.

Then

$$(46) \quad \text{ran}(\text{Id} - T_{(A,B)}) \simeq D \cap R.$$

Furthermore, the following implications hold:

$$(47) \quad (\exists C \in \{A, B\}) \text{ dom } C = X \text{ and } C \text{ is } 3^* \text{ monotone} \implies \text{ran}(\text{Id} - T_{(A,B)}) \simeq R,$$

$$(48) \quad (\exists C \in \{A, B\}) \text{ ran } C = X \text{ and } C \text{ is } 3^* \text{ monotone} \implies \text{ran}(\text{Id} - T_{(A,B)}) \simeq D,$$

and

$$(49) \quad \text{ri}(D \cap R) = D \cap R \implies \text{ran}(\text{Id} - T_{(A,B)}) = D \cap R.$$

Proof. First we show that

$$(50) \quad (\forall w \in \text{ri } D) \quad A + B_w \text{ is maximally monotone.}$$

Notice that $(\forall w \in X) \text{ dom } B_w = \text{dom } B + w$. Let $w \in \text{ri } D = \text{ri}(\text{dom } A - \text{dom } B)$. Then $\text{ri } \text{dom } A \cap \text{ri } \text{dom } B_w \neq \emptyset$. Using Fact 2.1, we conclude that $A + B_w$ is maximally monotone, which proves (50). Now, suppose that (i) holds. Then $(\forall w \in D \cap R) w \in \text{ri } \text{ran}(A + B_w) \subseteq \text{ran}(A + B_w)$. Combining with Fact 4.6 we conclude that $(\forall w \in \text{ri } D \cap \text{ri } R) w \in \text{ran}(\text{Id} - T_{(A,B)})$. Hence

$$(51) \quad \overline{\text{ri } D \cap \text{ri } R} \subseteq \overline{\text{ran}(\text{Id} - T_{(A,B)})}.$$

It follows from Lemma 5.1(v) that $\overline{\text{ri } D \cap \text{ri } R} = \overline{D \cap R}$. Altogether,

$$(52) \quad \overline{D \cap R} \subseteq \overline{\text{ran}(\text{Id} - T_{(A,B)})}.$$

It follows from (34) that $\overline{\text{ran}(\text{Id} - T_{(A,B)})} \subseteq \overline{D \cap R}$. Therefore,

$$(53) \quad \overline{D \cap R} = \overline{\text{ran}(\text{Id} - T_{(A,B)})}.$$

Since $T_{(A,B)}$ is firmly nonexpansive, hence nonexpansive, it follows from [5, Example 20.26] that $\text{Id} - T_{(A,B)}$ is maximally monotone, and therefore $\text{ran}(\text{Id} - T_{(A,B)})$ is nearly convex by Fact 3.2. Using Lemma 5.1(ii)&(iii) we know that $\text{ri } D \cap \text{ri } R \neq \emptyset$ and $D \cap R$ is nearly convex. Therefore, using (53) and Fact 3.5 applied to the nearly convex sets $D \cap R$ and $\text{ran}(\text{Id} - T_{(A,B)})$, we get $\text{ran}(\text{Id} - T_{(A,B)}) \simeq D \cap R$. Now we show that each of the conditions (ii) and (iii) imply (i). Let $w \in \text{ri } D \cap \text{ri } R$, and notice that (ii) implies that B_w is

3^* monotone, whereas (iii) implies that $\text{dom } B_w = \text{dom } B + w \subseteq \text{dom } A$. Using (50) and Fact 3.13 applied to A and B_w we have $(\forall w \in \text{ri } D \cap \text{ri } R)$

$$(54) \quad w \in \text{ri } R = \text{ri}(\text{ran } A + \text{ran } B) = \text{ri}(\text{ran } A + \text{ran } B_w) = \text{ri } \text{ran}(A + B_w).$$

That is, (i) holds, and consequently (46) holds.

We now turn to the implication (47). Observe first that $D = X$. If A is 3^* monotone and $\text{dom } A = X$, then clearly (iii) holds. Thus, it remains to consider the case when B is 3^* monotone and $\text{dom } B = X$. Then B_w is 3^* monotone and $\text{dom } A \subseteq X = \text{dom } B_w$. As before, we obtain $w \in \text{ri } R = \text{ri}(\text{ran } A + \text{ran } B) = \text{ri}(\text{ran } A + \text{ran } B_w) = \text{ri } \text{ran}(A + B_w)$. Hence (i) holds, which completes the proof of (47). To prove the implication (48), first notice that $(\exists C \in \{A, B\}) \text{ ran } C = X$ and C is 3^* monotone $\iff (\exists C \in \{A^{-1}, B^{-\odot}\}) \text{ dom } C = X$ and C is 3^* monotone. Therefore using Fact 4.2 and (47) applied to the operators A^{-1} and $B^{-\odot}$ $(\exists C \in \{A, B\}) \text{ ran } C = X$ and C is 3^* monotone $\implies \text{ran}(\text{Id} - T_{(A,B)}) = \text{ran}(\text{Id} - T_{(A^{-1}, B^{-\odot})}) = R_{(A^{-1}, B^{-\odot})} = \text{ran } A^{-1} + \text{ran } B^{-\odot} = \text{dom } A - \text{dom } B = D$, which proves (48).

Now suppose that $\text{ri}(D \cap R) = D \cap R$. It follows from (46) and (34) that

$$(55) \quad D \cap R = \text{ri}(D \cap R) = \text{ri } \text{ran}(\text{Id} - T_{(A,B)}) \subseteq \text{ran}(\text{Id} - T_{(A,B)}) \subseteq D \cap R.$$

Hence all the inclusions become equalities, which proves (49). ■

Corollary 5.3 (range of the Douglas–Rachford operator) *Suppose that A and B satisfy one of the following:*

- (i) $(\forall w \in \text{ri } D_{(A,B^{-1})} \cap \text{ri } R_{(A,B^{-1})}) \quad \text{ri}(\text{ran } A + \text{dom } B) \subseteq \text{ri } \text{ran}(A + B_w^{-1}).$
- (ii) A and B are 3^* monotone.
- (iii) $\text{ran } B + \text{ri } D_{(A,B^{-1})} \cap \text{ri } R_{(A,B^{-1})} \subseteq \text{dom } A$ and A is 3^* monotone.

Then

$$(56) \quad \text{ran } T_{(A,B)} \simeq (\text{dom } A - \text{ran } B) \cap (\text{ran } A + \text{dom } B).$$

Furthermore, the following implications hold:

$$(57) \quad (\exists C \in \{A, B^{-1}\}) \quad \text{dom } C = X \text{ and } C \text{ is } 3^* \text{ monotone} \implies \text{ran } T_{(A,B)} \simeq \text{ran } A + \text{dom } B$$

and

$$(58) \quad \begin{aligned} \text{ri}(D_{(A,B^{-1})} \cap R_{(A,B^{-1})}) &= D_{(A,B^{-1})} \cap R_{(A,B^{-1})} \\ &\implies \text{ran } T_{(A,B)} = (\text{dom } A - \text{ran } B) \cap (\text{ran } A + \text{dom } B). \end{aligned}$$

Proof. Using Fact 4.4, we know that $T_{(A,B)} = \text{Id} - T_{(A,B^{-1})}$. The result thus follows by applying Theorem 5.2 to (A, B^{-1}) . ■

The assumptions in Theorem 5.2 are critical. Example 4.9 shows that when neither A nor B is 3^* monotone, the conclusion of the theorem fails. Now we show that the conclusion of Theorem 5.2 fails even if one of the operators is a subdifferential operator.

Example 5.4 Suppose that $X = \mathbb{R}^2$, set $C = \mathbb{R} \times \{0\}$, and suppose that

$$(59) \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = N_C.$$

Then $\text{Id} - T_{(A,B)} = J_A - P_C R_A$. Notice that $P_C : (x, y) \mapsto (x, 0)$, $J_A : (x, y) \mapsto \frac{1}{2}(x + y, -x + y)$ and consequently $R_A : (x, y) \mapsto (y, -x)$. Hence

$$(60) \quad \text{ran}(\text{Id} - T_{(A,B)}) = \mathbb{R} \cdot (1, -1) \subsetneq \mathbb{R}^2 = (\text{dom } A - \text{dom } B) \cap (\text{ran } B + \text{ran } A).$$

Corollary 5.5 Suppose that A and B satisfy one of the following:

- (i) $(\forall w \in \text{ri } D \cap \text{ri } R) \quad \text{ri}(\text{ran } A + \text{ran } B) \subseteq \text{ri } \text{ran}(A + B_w).$
- (ii) A and B are 3^* monotone.
- (iii) $\text{dom } B + \text{ri } D \cap \text{ri } R \subseteq \text{dom } A$ and A is 3^* monotone.
- (iv) $(\exists C \in \{A, B\}) \text{ dom } C = X$ and C is 3^* monotone.

Furthermore, suppose that D and R are affine subspaces. Then $\text{ran}(\text{Id} - T_{(A,B)}) = D \cap R$.

Proof. Since $\text{ri } D = D$ and $\text{ri } R = R$, Lemma 5.1(iv) yields $D \cap R = \text{ri } D \cap \text{ri } R = \text{ri}(D \cap R)$. Now apply (49). ■

Corollary 5.6 Suppose that $X = \mathbb{R}$. Then $\text{ran}(\text{Id} - T_{(A,B)}) \simeq D \cap R$.

Proof. Indeed, it follows from e.g. [5, Corollary 22.19] and Fact 3.12 that A and B are 3^* monotone. Now apply Theorem 5.2(ii). ■

We now construct an example where $\text{ran}(\text{Id} - T_{(A,B)})$ properly lies between $\text{ri}(D \cap R)$ and $\overline{D \cap R}$. This illustrates that Theorem 5.2 is optimal in the sense that near equality cannot be replaced by actual equality.

Example 5.7 Suppose that $\dim X \geq 2$, let u and v be in X with $u \neq v$, let r and s be in \mathbb{R}_{++} , set $U = \text{ball}(u; r)$ and $V = \text{ball}(v; s)$, and suppose that $A = N_U$ and $B = N_V$. Then $D \cap R = \text{ball}(u - v; r + s)$ and

(61)

$$\text{ran}(\text{Id} - T_{(A,B)}) = \text{int ball}(u - v; r + s) \cup \left\{ \left(1 - \frac{r+s}{\|u-v\|}\right) (u - v), \left(1 + \frac{r+s}{\|u-v\|}\right) (u - v) \right\};$$

consequently,

(62)

$$\text{ri}(D \cap R) \subsetneq \text{ran}(\text{Id} - T_{(A,B)}) \subsetneq \overline{D \cap R}.$$

Moreover,

(63)

$$v_{(A,B)} = \max \{ (r + s) - \|v - u\|, 0 \} \cdot \frac{v - u}{\|v - u\|}.$$

Proof. It follows from Fact 3.12 that A and B are 3^* monotone. Using e.g. [5, Corollary 21.21], we have $\text{ran } A = \text{ran } B = X$, hence $R = X$ and $D \cap R = D = U - V$. First notice that

(64)

$$D \cap R = D = U - V = \text{ball}(u - v; r + s).$$

We claim that

(65)

$$(\forall w \in D \setminus \text{ri } D) \quad U \cap (V + w) \quad \text{is a singleton.}$$

Since $D = U - V$, we have $(\forall w \in D) \quad U \cap (V + w) \neq \emptyset$. Now let $w \in D \setminus \text{ri } D$ and assume to the contrary that $\{y, z\} \subseteq U \cap (V + w)$ with $y \neq z$. Then $\{y - w, z - w\} \subseteq V$, and $(\forall \lambda \in]0, 1[)$

(66)

$$\lambda y + (1 - \lambda)z \in \text{int } U \quad \text{and} \quad \lambda y + (1 - \lambda)z - w \in \text{int } V.$$

It follows from Fact 3.6 and the above inclusions that $w \in \text{int } U - \text{int } V = \text{ri } U - \text{ri } V = \text{ri } D$, which is absurd. Therefore (65) holds. Now, let $w \in D \setminus \text{ri } D$ and notice that $V + w = \text{ball}(v + w; r)$. Using (65) we have $U \cap (V + w) = \text{dom}(A + B_w)$ is a singleton. Consequently, $\text{ran}(A + B_w)$ is the line passing through the origin parallel to the line passing through u and $v + w$, and by Fact 4.6, we have $w \in \text{ran}(\text{Id} - T_{(A,B)}) \iff w \in \text{ran}(A + B_w) \iff w = \lambda(u - v - w)$ for some $\lambda \in \mathbb{R} \setminus \{-1\} \iff w = \frac{\lambda}{1+\lambda}(u - v)$ with $\lambda \in \mathbb{R} \setminus \{-1\}$ (since $u \neq v$), or equivalently,

(67)

$$w = \alpha(u - v), \quad \text{where } \alpha \in \mathbb{R} \setminus \{1\}.$$

Finally notice that w is on the boundary of $U - V$. Therefore, using (64) and (67) we must have $\|w - (u - v)\| = r + s \iff |\alpha - 1|\|u - v\| = r + s \iff \alpha = 1 \pm \frac{r+s}{\|u-v\|}$, which means that only two points on the boundary of D are included in $\text{ran}(\text{Id} - T_{(A,B)})$. Moreover, if $\|u - v\| > r + s$, then $0 \in \text{int ball}(u - v; r + s)$, hence $v_{(A,B)} = 0$. Else if $\|u - v\| \leq r + s$, using [5, Proposition 28.10] we get $v_{(A,B)} = \left(1 - \frac{r+s}{\|u-v\|}\right) (u - v)$, which completes the proof. \blacksquare

6 Applications

6.1 On the infimal displacement vector $v_{(A,B)}$

In this section, we focus on $v_{(A,B)}$.

Proposition 6.1 *Suppose that A and B satisfy one of the following:*

- (i) $(\forall w \in \text{ri } D \cap \text{ri } R) \quad \text{ri}(\text{ran } A + \text{ran } B) \subseteq \text{ri } \text{ran } (A + B_w).$
- (ii) A and B are 3^* monotone.
- (iii) $\text{dom } B + \text{ri } D \cap \text{ri } R \subseteq \text{dom } A$ and A is 3^* monotone.
- (iv) $(\exists C \in \{A, B\}) \text{ dom } C = X$ and C is 3^* monotone.

Then $\overline{\text{ran}}(\text{Id} - T_{(A,B)}) = \overline{D \cap R} = \overline{D} \cap \overline{R}$ and $v_{(A,B)} = P_{\overline{D \cap R}}(0).$

Proof. Combine Theorem 5.2, Lemma 5.1(vi), and (39). ■

Using the symmetric hypotheses of Theorem 5.2, we obtain the following result:

Lemma 6.2 *Suppose that both A and B are 3^* monotone, or that $(\exists C \in \{A, B\})$ such that $\text{dom } C = X$ and C is 3^* monotone. Then the following hold:*

- (i) *If D is a linear subspace of X , then $\text{ran}(\text{Id} - T_{(A,B)}) \simeq \text{ran}(\text{Id} - T_{(B,A)})$ and $v_{(A,B)} = v_{(B,A)}.$*
- (ii) *If R is a linear subspace of X , then $\text{ran}(\text{Id} - T_{(A,B)}) \simeq -\text{ran}(\text{Id} - T_{(B,A)})$ and $v_{(A,B)} = -v_{(B,A)}.$*
- (iii) *If $\text{dom } A = X$ or $\text{dom } B = X$, then $\text{ran}(\text{Id} - T_{(A,B)}) \simeq \text{ran}(\text{Id} - T_{(B,A)}) \simeq R$, and $v_{(A,B)} = v_{(B,A)} = P_{\overline{R}}(0).$*
- (iv) *If $\text{dom } A$ or $\text{dom } B$ is bounded, then $\text{ran}(\text{Id} - T_{(A,B)}) \simeq -\text{ran}(\text{Id} - T_{(B,A)}) \simeq D$, and $v_{(A,B)} = -v_{(B,A)} = P_{\overline{D}}(0).$*

Proof. Observe first that

$$(68) \quad \text{ran}(\text{Id} - T_{(B,A)}) \simeq (-D) \cap R$$

by (44). (i): Since $D = -D$, Theorem 5.2 and (68) yield $\text{ran}(\text{Id} - T_{(A,B)}) \simeq \text{ran}(\text{Id} - T_{(B,A)})$, and the conclusion follows from (39). (ii): Let $u \in X$. Since $R = -R$, we obtain the equivalences $u \in D \cap R \iff -u \in -D \text{ and } -u \in R \iff -u \in (-D) \cap R \iff u \in -((-D) \cap R)$. Hence, $D \cap R = -((-D) \cap R)$. Consequently, $\overline{D \cap R} = -\overline{((-D) \cap R)}$ and $\text{ri}(D \cap R) = -\text{ri}((-D) \cap R)$. Applying Theorem 5.2, in view of (68), to the pair (A, B) and the pair (B, A) , we conclude $\text{ran}(\text{Id} - T_{(A,B)}) \simeq -\text{ran}(\text{Id} - T_{(B,A)})$. Thus $\overline{\text{ran}}(\text{Id} - T_{(A,B)}) = -\overline{\text{ran}}(\text{Id} - T_{(B,A)})$, and the result follows from (39). (iii): The hypothesis implies $D = X = \overline{X} = \overline{D}$. Now combine with (i) and Proposition 6.1(ii). (iv): That either $\text{dom } A$ or $\text{dom } B$ is bounded, implies that $\text{ran } A = X$ (respectively $\text{ran } B = X$) (see, e.g., [5, Corollary 21.21]). Hence $R = X = \overline{X} = \overline{R}$. Now combine with (ii) and Proposition 6.1(ii). ■

Example 6.3 Suppose that $X = \mathbb{R}$. It follows from Fact 4.8 that $v_{(A,B)} = \pm v_{(B,A)}$.

In [2, Section 3], we constructed examples where

$$(69) \quad \frac{\langle v_{(A,B)}, v_{(B,A)} \rangle}{\|v_{(A,B)}\| \|v_{(B,A)}\|} \in \{-1, 0, 1\}.$$

We now show that this quotient can take on any value in $[-1, 1]$.

Example 6.4 (angle between $v_{(A,B)}$ and $v_{(B,A)}$) Suppose that S is a linear subspace of X such that $\{0\} \subsetneq S \subsetneq X$. Let $\theta \in \mathbb{R}$, let $u \in S$, and let $v \in S^\perp$ such that $\|u\| = \|v\| = 1$. Set $a = \sin(\theta)v$, and set $b = \cos(\theta)u$. Suppose that $A = N_{S+a}$ and that $B = N_S + b$. Then $D = \text{dom } A - \text{dom } B = S + a - S = S + a$, and $R = \text{ran } A + \text{ran } B = S^\perp + S^\perp + b = S^\perp + b$. Consequently, $-D = S - a$. Clearly, $\overline{D \cap R} = D \cap R = \{b + a\}$, whereas $\overline{(-D) \cap R} = (-D) \cap R = \{b - a\}$. Therefore, $v_{(A,B)} = b + a$, and $v_{(B,A)} = b - a$. By Fact 4.8 $\|v_{(A,B)}\| = \|v_{(B,A)}\| = 1$. Moreover, since $a \perp b$

$$(70) \quad \langle v_{(A,B)}, v_{(B,A)} \rangle = \langle b + a, b - a \rangle = \|b\|^2 - \|a\|^2 = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta).$$

6.2 Subdifferential operators

We now turn to subdifferential operators.

Corollary 6.5 Let $f : X \rightarrow]-\infty, +\infty]$ and $g : X \rightarrow]-\infty, +\infty]$ be proper lower semicontinuous convex functions. Then the following hold:

- (i) $\text{ran}(\text{Id} - T_{(\partial f, \partial g)}) \simeq (\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*)$.

$$(ii) \operatorname{ran} T_{(\partial f, \partial g)} \simeq (\operatorname{dom} f - \operatorname{dom} g^*) \cap (\operatorname{dom} f^* + \operatorname{dom} g).$$

Proof. It is well-known that (see, e.g., [5, Corollary 16.29]) $\overline{\operatorname{dom} f} = \overline{\operatorname{dom} \partial f}$. Since f is convex, so is $\operatorname{dom} f$. Moreover, by Fact 3.2 $\operatorname{dom} \partial f$ is nearly convex. Therefore, applying Fact 3.5 to the sets $\operatorname{dom} f$ and $\operatorname{dom} \partial f$ we conclude that $\operatorname{dom} \partial f \simeq \operatorname{dom} f$. Using [5, Proposition 16.24], and the previous conclusion applied to f^* , we have $\operatorname{ran} \partial f = \operatorname{dom}(\partial f)^{-1} = \operatorname{dom} \partial f^* \simeq \operatorname{dom} f^*$. Altogether,

$$(71) \quad \operatorname{dom} \partial f \simeq \operatorname{dom} f, \quad \operatorname{ran} \partial f \simeq \operatorname{dom} f^*.$$

Applying Fact 3.7 with $C_1 = \operatorname{dom} f$, $C_2 = -\operatorname{dom} g$, $D_1 = \operatorname{dom} \partial f$, $D_2 = -\operatorname{dom} \partial g$, we conclude that

$$(72) \quad \operatorname{dom} \partial f - \operatorname{dom} \partial g \simeq \operatorname{dom} f - \operatorname{dom} g.$$

One shows similarly that

$$(73) \quad \operatorname{ran} \partial f + \operatorname{ran} \partial g \simeq \operatorname{dom} f^* + \operatorname{dom} g^*.$$

It follows from the maximal monotonicity of ∂f and ∂g and Lemma 5.1(ii) that $(\operatorname{dom} \partial f - \operatorname{dom} \partial g) \cap (\operatorname{ran} \partial f + \operatorname{ran} \partial g) \neq \emptyset$. Applying Corollary 3.10 with $C_1 := \operatorname{dom} \partial f - \operatorname{dom} \partial g$, $C_2 := \operatorname{ran} \partial f + \operatorname{ran} \partial g$, $D_1 := \operatorname{dom} f - \operatorname{dom} g$, and $D_2 := \operatorname{dom} f^* + \operatorname{dom} g^*$, we conclude that

$$(74) \quad (\operatorname{dom} \partial f - \operatorname{dom} \partial g) \cap (\operatorname{ran} \partial f + \operatorname{ran} \partial g) \simeq (\operatorname{dom} f - \operatorname{dom} g) \cap (\operatorname{dom} f^* + \operatorname{dom} g^*).$$

To complete the proof, notice that by Fact 3.12 ∂f and ∂g are 3^* monotone operators, and by assumption $\partial f + \partial g$ is maximally monotone. Therefore, by Theorem 5.2, we have

$$(75) \quad \operatorname{ran}(\operatorname{Id} - T_{(\partial f, \partial g)}) \simeq (\operatorname{dom} \partial f - \operatorname{dom} \partial g) \cap (\operatorname{ran} \partial f + \operatorname{ran} \partial g).$$

Combining (74) and (75) we conclude that (i) holds true. To prove (ii), combine Corollary 5.3, (71) and Corollary 3.10. ■

Corollary 6.6 *Let $f : X \rightarrow]-\infty, +\infty]$ be proper, convex, lower semicontinuous and suppose that V is a nonempty closed convex subset of X . Suppose that $A = \partial f$ and $B = N_V$. Then the following hold:*

$$(i) \quad T_{(A,B)} = J_{N_V} R_{\partial f} + \operatorname{Id} - J_{\partial f} = P_V(2 \operatorname{Prox}_f - \operatorname{Id}) + \operatorname{Id} - \operatorname{Prox}_f.$$

$$(ii) \quad \operatorname{ran}(\operatorname{Id} - T_{(A,B)}) \simeq (\operatorname{dom} f - V) \cap (\operatorname{dom} f^* + (\operatorname{rec} V)^\ominus).$$

$$(iii) \quad \operatorname{ran} T_{(A,B)} \simeq (\operatorname{dom} f - (\operatorname{rec} V)^\ominus) \cap (\operatorname{dom} f^* + V).$$

Consequently, if V is a linear subspace we may add to this list the following items:

$$(iv) \text{ ran}(\text{Id} - T_{(A,B)}) \simeq (\text{dom } f + V) \cap (\text{dom } f^* + V^\perp).$$

$$(v) \text{ ran } T_{(A,B)} \simeq (\text{dom } f + V^\perp) \cap (\text{dom } f^* + V).$$

Proof. Since $\text{ran } N_V$ is nearly convex and $(\text{rec } V)^\ominus$ is convex, it follows from (8), Fact 2.4 and Fact 3.5 that

$$(76) \quad \text{ran } N_V \simeq (\text{rec } V)^\ominus.$$

(i): This follows from (32) and the fact that $J_{N_V} = P_V$ and $J_{\partial f} = \text{Prox } f$. (ii): Combine (71), (76), Theorem 5.2, Fact 3.7 and Corollary 3.10. (iii): Combine (71), (76), Corollary 5.3, Fact 3.7 and Corollary 3.10. (iv) and (v): It follows from [5, Proposition 6.22 and Corollary 6.49] that $\text{rec } V = V$ and $(\text{rec } V)^\ominus = V^\perp$. Combining this with (ii) and (iii), we obtain (iv) and (v), respectively. ■

Corollary 6.7 (two normal cone operators) *Let U and V be two nonempty closed convex subsets of X , and suppose that $A = N_U$ and that $B = N_V$. Then the following hold:*

$$(i) \text{ ran}(\text{Id} - T_{(A,B)}) \simeq (U - V) \cap ((\text{rec } U)^\ominus + (\text{rec } V)^\ominus).$$

$$(ii) \text{ ran } T_{(A,B)} \simeq (U - (\text{rec } V)^\ominus) \cap ((\text{rec } U)^\ominus + V).$$

$$(iii) v_{(A,B)} = P_{\overline{U-V}}(0).$$

Proof. Clearly $\text{dom } A = U$ and $\text{dom } B = V$. It follows from (76) that $\text{ran } N_U \simeq (\text{rec } U)^\ominus$ and $\text{ran } N_V \simeq (\text{rec } V)^\ominus$. Therefore, Fact 3.7 implies that

$$(77) \quad R \simeq (\text{rec } U)^\ominus + (\text{rec } V)^\ominus.$$

Now (i) follows from combining (77) and Theorem 5.2, and (ii) follows from combining (76) applied to the sets U and V , Fact 3.7 and Corollary 5.3. It remains to show (iii) is true. Set $v = P_{\overline{U-V}}(0) = P_{\overline{D}}(0)$. On the one hand, by definition of v and Proposition 6.1(ii), we have $v_{(A,B)} \in \overline{D} \cap \overline{R} \subseteq \overline{D}$ and hence

$$(78) \quad \|v\| \leq \|v_{(A,B)}\|.$$

On the other hand, using [6, Corollary 2.7] we have $v \in \overline{(P_U - \text{Id})(V)} \cap \overline{(\text{Id} - P_V)(U)} \subseteq (\text{rec } U)^\oplus \cap (\text{rec } V)^\ominus$. Therefore, using (76) and that $0 \in (\text{rec } U)^\ominus$ we have $v \in (\text{rec } U)^\oplus \cap (\text{rec } V)^\ominus \subseteq (\text{rec } V)^\ominus \subseteq (\text{rec } U)^\ominus + (\text{rec } V)^\ominus \subseteq \overline{R}$. Hence,

$$(79) \quad v \in \overline{D} \cap \overline{R}.$$

Combining (78), (79) and Proposition 6.1(ii) yields $v = v_{(A,B)}$. ■

6.3 Firmly nonexpansive mappings

We now restate the main result from the perspective of fixed point theory.

Corollary 6.8 *Let $T_1 : X \rightarrow X$ and $T_2 : X \rightarrow X$ be firmly nonexpansive such that each T_i satisfies*

$$(80) \quad (\forall x \in X)(\forall y \in X) \quad \inf_{z \in X} \langle T_1 x - T_1 z, (y - T_1 y) - (z - T_1 z) \rangle > +\infty,$$

and set $T := T_2(2T_1 - \text{Id}) + \text{Id} - T_1$. Then

$$(81) \quad \text{ran } T \simeq (\text{ran } T_1 - \text{ran}(\text{Id} - T_2)) \cap (\text{ran}(\text{Id} - T_1) + \text{ran } T_2),$$

and

$$(82) \quad \text{ran}(\text{Id} - T) \simeq (\text{ran } T_1 - \text{ran } T_2) \cap (\text{ran}(\text{Id} - T_1) + \text{ran}(\text{Id} - T_2)).$$

Proof. Using Fact 2.5 we conclude that there exist maximally monotone operators $A : X \rightrightarrows X$ and $B : X \rightrightarrows X$ such that

$$(83) \quad T_1 = J_A \quad \text{and} \quad T_2 = J_B.$$

Moreover, it follows from [7, Theorem 2.1(xvii)] and (80) that that A and B are 3^* monotone. By (31), we conclude that $T = T_{(A,B)}$. Using Corollary 5.3, Fact 2.2 and (8) we have

$$\begin{aligned} \text{ran } T &\simeq (\text{dom } A - \text{ran } B) \cap (\text{ran } A + \text{dom } B) \\ &= (\text{ran}(\text{Id} - T_1) - \text{ran } T_2) \cap (\text{ran } T_1 + \text{ran}(\text{Id} - T_2)). \end{aligned}$$

That is, (81) holds true. Similarly, one can prove (82) by combining Theorem 5.2, Fact 2.2 and (8). ■

7 Some infinite-dimensional observations

In this final section, we provide some results that remain true in infinite-dimensional settings. We assume henceforth that

$$(84) \quad H \text{ is a (possibly infinite-dimensional) real Hilbert space.}$$

A pleasing identity arises when the we are dealing with normal cone operators of closed subspaces.

Proposition 7.1 Let U and V be closed linear subspaces of H , and suppose that $A = N_U$ and $B = N_V$. Then $\text{ran}(\text{Id} - T_{(A,B)}) = (U + V) \cap (U^\perp + V^\perp)$.

Proof. Since $\text{gra } N_U = U \times U^\perp$ and $\text{gra } N_V = V \times V^\perp$, the result follows from (33). \blacksquare

Proposition 7.2 Let U and V be closed linear subspaces of H such that

$$(85) \quad U^\perp \cap V = \{0\},$$

and suppose that $A = N_U$ and $B = P_V$. Then the following hold:

- (i) $U^\perp \cap P_V^{-1}(P_V(U^\perp) \setminus P_V(U)) \subseteq (D \cap R) \setminus \text{ran}(\text{Id} - T_{(A,B)})$.
- (ii) $(U^\perp + V) \cap (V + P_V^{-1}(P_V(U^\perp) \setminus P_V(U))) \subseteq (\text{ran } A + \text{ran } B) \setminus \text{ran}(A + B)$.

Consequently, if $P_V(U^\perp) \setminus P_V(U) \neq \emptyset$, then

$$(86) \quad \text{ran}(\text{Id} - T_{(A,B)}) \subsetneq D \cap R$$

and

$$(87) \quad \text{ran}(A + B) \subsetneq \text{ran } A + \text{ran } B.$$

Proof. It is clear that $D \cap R = (U - H) \cap (U^\perp + V) = U^\perp + V$. Notice that (i) and (ii) trivially hold when $P_V(U^\perp) \setminus P_V(U) = \emptyset$. Now suppose that $P_V(U^\perp) \setminus P_V(U) \neq \emptyset$. It follows from (37) and (85) that $(\forall w \in U^\perp \subseteq U^\perp + V)$

$$(88) \quad \begin{aligned} Z_w \neq \emptyset &\iff (\exists u \in U) \text{ such that } w \in N_U u + P_V(u - w) \\ &\iff (\exists u \in U) P_V u - P_V w \in U^\perp - w = U^\perp \\ &\iff (\exists u \in U) P_V u - P_V w = 0 \iff P_V w \in P_V(U). \end{aligned}$$

Now let $w \in U^\perp \subseteq U^\perp + V = D \cap R$ such that $P_V w \notin P_V(U)$. Then (88) implies that $Z_w = \emptyset$, hence by (38) $w \notin \text{ran}(\text{Id} - T_{(A,B)})$, which proves (i) and consequently (86). To complete the proof we need to show that (ii) holds. Notice that $(\forall u^\perp \in U^\perp)$ $u^\perp + P_V u^\perp \in U^\perp + V = \text{ran } A + \text{ran } B$. It follows from (85) that

$$(89) \quad \begin{aligned} u^\perp + P_V u^\perp \in \text{ran}(A + B) &\iff (\exists u \in U = \text{dom}(A + B)) u^\perp + P_V u^\perp \in U^\perp + P_V u \\ &\iff (\exists u \in U) P_V u^\perp - P_V u \in U^\perp - u^\perp = U^\perp \\ &\iff (\exists u \in U) P_V u^\perp = P_V u. \end{aligned}$$

Now, let $u^\perp \in U^\perp$ such that $P_V u^\perp \notin P_V(U)$. Then using (89) $w = u^\perp + P_V u^\perp \notin \text{ran}(A + B)$. Notice that by construction $w \in U^\perp + V = \text{ran } A + \text{ran } B$. \blacksquare

Remark 7.3 Notice that in Proposition 7.2 both A and B are linear relations, maximally monotone and 3^* monotone operators. Consequently, the sets D and R are linear subspaces of H . When H is finite-dimensional, Corollary 5.5 and [11, footnote on page 174] imply that $\text{ran}(\text{Id} - T) = D \cap R$ and $\text{ran}(A + B) = \text{ran } A + \text{ran } B$. Thus, if (86) or (87) holds, then H is necessarily infinite-dimensional.

We now provide a concrete example in $\ell^2(\mathbb{N})$ where both (86) and (87) hold. This illustrates again the requirement of the closure in Fact 3.13.

Proposition 7.4 Suppose that $H = \ell^2(\mathbb{N})$, let $p \in \mathbb{R}_{++}$, and let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R}_{++} such that

$$(90) \quad \alpha_n \rightarrow 0, \quad \sum_{n=0}^{\infty} \alpha_n^{2p-2} < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n^{2p-4} = +\infty.$$

Set $U = \{x = (x_n)_{n \in \mathbb{N}} \in H \mid x_{2n+1} = -\alpha_n x_{2n}\}$ and $V = \{x = (x_n)_{n \in \mathbb{N}} \in H \mid x_{2n} = 0\}$, and suppose that $A = N_U$ and $B = P_V$. Then $P_V(U^\perp) \setminus P_V(U) \neq \emptyset$ and hence

$$(91) \quad \text{ran}(\text{Id} - T_{(A,B)}) \subsetneq D \cap R \quad \text{and} \quad \text{ran}(A + B) \subsetneq \text{ran } A + \text{ran } B.$$

Proof. It is easy to check that $U^\perp = \{x = (x_n) \in H \mid x_{2n+1} = \alpha_n^{-1} x_{2n}\}$. Hence $U^\perp \cap V = \{0\}$. Let $w \in H$ be defined as $(\forall n \in \mathbb{N}) w_{2n} = \alpha_n^p$ and $w_{2n+1} = \alpha_n^{p-1}$. Clearly $w \in U^\perp$. We claim that $P_V w \notin P_V U$. Suppose this is not true. Then $(\exists u \in U)$ such that $P_V w = P_V u$. Hence $(\forall n \in \mathbb{N}) u_{2n+1} = (P_V u)_{2n+1} = (P_V w)_{2n+1} = w_{2n+1} = \alpha_n^{p-1}$. Consequently, $(\forall n \in \mathbb{N}) u_{2n} = -\alpha_n^{p-2}$, which is absurd since it implies that $\sum_{n=0}^{\infty} u_{2n}^2 = \sum_{n=0}^{\infty} \alpha_n^{2p-4} = +\infty$, by (90). Therefore, $P_V(U^\perp) \setminus P_V(U) \neq \emptyset$. Using Proposition 7.2 we conclude that (91) holds. \blacksquare

The next example is a special case of Proposition 7.4.

Example 7.5 Suppose that $H = \ell^2(\mathbb{N})$, let $(\alpha_n)_{n \in \mathbb{N}} = (1/(n+1))_{n \in \mathbb{N}}$, let $p \in]\frac{3}{2}, \frac{5}{2}]$ and let U, V, A and B be as defined in Proposition 7.4. Since $2p - 2 > 1$ and $2p - 4 \leq 1$, we see that (90) holds. From Proposition 7.4 we conclude that $\text{ran}(\text{Id} - T_{(A,B)}) \subsetneq D \cap R$ and $\text{ran}(A + B) \subsetneq \text{ran } A + \text{ran } B$.

When A or B has additional structure, it may be possible to traverse between $\text{ran}(A + B)$ and $\text{ran}(\text{Id} - T_{(A,B)})$ as we illustrate now.

Proposition 7.6 Let $A : H \rightrightarrows H$ and $B : H \rightrightarrows H$ be maximally monotone. Then the following hold:

- (i) If $B : H \rightarrow H$ is linear, then $\text{ran}(\text{Id} - T_{(A,B)}) = J_B(\text{ran}(A + B))$ and $\text{ran}(A + B) = (\text{Id} + B) \text{ran}(\text{Id} - T_{(A,B)})$.
- (ii) If $A : H \rightarrow H$ is linear and $\text{Id} - A$ is invertible, then $\text{ran}(\text{Id} - T_{(A,B)}) = (\text{Id} - A)^{-1}(\text{ran}(A + B))$ and $\text{ran}(A + B) = (\text{Id} - A) \text{ran}(\text{Id} - T_{(A,B)})$.
- (iii) If $A : H \rightarrow H$ and $B : H \rightarrow H$ are linear and $A^* = -A$, then $(\forall \lambda \in [0, 1])$ $\text{ran}(\text{Id} - T_{(A,B)}) = J_{\lambda A^* + (1-\lambda)B}(\text{ran}(A + B))$.

Proof. Let $w \in X$. It follows from (38) that

$$(92) \quad w \in \text{ran}(\text{Id} - T_{(A,B)}) \iff (\exists x \in H) \text{ such that } w \in Ax + B(x - w).$$

(i): It follows from (92) that $w \in \text{ran}(\text{Id} - T_{(A,B)}) \iff (\exists x \in H) \text{ such that } w \in Ax + Bx - Bw \iff (\exists x \in H) (\text{Id} + B)w = w + Bw \in (A + B)x \iff (\exists x \in H) w \in J_B((A + B)x) \iff w \in J_B(\text{ran}(A + B))$. Using [7, Theorem 2.1(ii)&(iv)] we learn that J_B is a bijection, hence invertible, and $\text{ran}(A + B) = J_B^{-1} \text{ran}(\text{Id} - T_{(A,B)}) = (\text{Id} + B) \text{ran}(\text{Id} - T_{(A,B)})$, as claimed. (ii): It follows from (92) that $w \in \text{ran}(\text{Id} - T_{(A,B)}) \iff (\exists x \in H) \text{ such that } w - Aw \in A(x - w) + B(x - w) = (A + B)(x - w) \iff (\exists x \in H) (\text{Id} - A)w \in (A + B)(x - w) \iff (\exists x \in H) w \in (\text{Id} - A)^{-1}((A + B)(x - w)) \iff w \in (\text{Id} - A)^{-1}(\text{ran}(A + B))$. Since $\text{Id} - A$ is invertible, we learn that $\text{Id} - A$ is a bijection and $\text{ran}(A + B) = (\text{Id} - A) \text{ran}(\text{Id} - T_{(A,B)})$, as claimed. (iii): It follows from (92) that $w \in \text{ran}(\text{Id} - T_{(A,B)}) \iff (\exists x \in H) \text{ such that } (\forall \lambda \in [0, 1]) w - \lambda Aw + (1 - \lambda)Bw = A(x - \lambda w) + B(x - w + (1 - \lambda)w) \in (A + B)(x - \lambda w) \iff (\text{Id} + \lambda A^* + (1 - \lambda)B)w \in \text{ran}(A + B) \iff w \in J_{\lambda A^* + (1-\lambda)B}(\text{ran}(A + B))$. ■

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